horizontal asymptote - HA
vertical asymptote - VA

point of discontinuity → PD

Assignment

1. For each of the graphs of the following rational functions, algebraically determine the equation of any asymptotes and the coordinates of any points of discontinuity.

a) $f(x) = \dfrac{2(x+1)}{x^2+1}$

horizontal asymptote
$y = 0$

b) $g(x) = -\dfrac{4x^3}{x^3+1}$ $x \neq -1$

$x^3+1=0$
$x^3=-1$

HA $y=-4$
VA $x=-1$

c) $h(x) = \dfrac{3-x^4}{3x^4+6} = \dfrac{-x^4+3}{3x^4+6}$

HA $y = -\frac{1}{3}$

d) $k(x) = \dfrac{x+4}{x^2+5x+4} = \dfrac{x+4}{(x+1)(x+4)}$

$= \dfrac{1}{x+1}$, $x \neq -4$

$\dfrac{1}{-4+1} = \dfrac{1}{-3}$

HA $y=0$
VA $x=-1$
PD $(-4, -1/3)$

e) $f(x) = \dfrac{2(3x-1)(x+4)}{3x^2+4x+1}$

$3x^2+4x+1$
$= 3x^2+3x+x+1$
$= 3x(x+1)+1(x+1)$
$= (x+1)(3x+1)$

$f(x) = \dfrac{2(3x-1)(x+4)}{(x+1)(3x+1)}$

HA $y=2$
VA $x=-1, -1/3$

f) $f(x) = \dfrac{2(3x-1)(x+4)}{3x^2+10x-8}$

$3x^2+10x-8$
$= 3x^2-2x+12x-8$
$= x(3x-2)+4(3x-2)$
$= (x+4)(3x-2)$

$f(x) = \dfrac{2(3x-1)(x+4)}{(x+4)(3x-2)} = \dfrac{2(3x-1)}{3x-2}$, $x \neq$

if $x=-4$, $\dfrac{2(3(4)-1)}{3(4)-2} = \dfrac{2(-13)}{-14}$
$= \dfrac{13}{7}$

HA $y=2$
VA $\cancel{x} = 2/3$
PD $(-4, 13/7)$

2. Consider the functions $f(x) = x^2 - 9$ and $g(x) = 9x - x^3$.

a) Express $f(x)$ and $g(x)$ in factored form.

$f(x) = x^2 - 9 = (x-3)(x+3)$

$g(x) = 9x - x^3 = x(9-x^2) = x(3-x)(3+x)$

b) Express the function $\left(\frac{f}{g}\right)(x)$ in simplest form.

$\left(\frac{f}{g}\right)(x) = \frac{x^2-9}{9x-x^3} = \frac{(x-3)(x+3)}{x(3-x)(\cancel{3+x})} = -\frac{1}{x}, \ x \neq \pm 3.$

c) Algebraically determine the equation of any asymptotes and the coordinates of any points of discontinuity on the graph of $y = \left(\frac{f}{g}\right)(x)$.

if $x = -3$ $\quad -\frac{1}{x} = \frac{-1}{-3} = \frac{1}{3}$

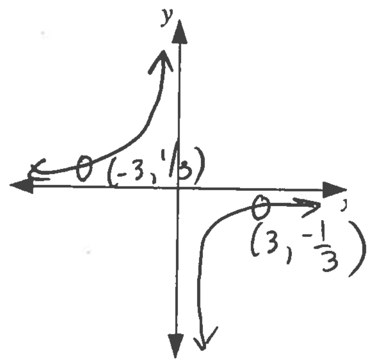
if $x = +3$ $\quad -\frac{1}{x} = \frac{-1}{3} = \frac{-1}{3}$

HA $y = 0$

VA $x = 0$

PD $(-3, \frac{1}{3})$ + $(3, \frac{-1}{3})$

d) Sketch a graph of $y = \left(\frac{f}{g}\right)(x)$ on the grid illustrating the features in c).



e) Determine the domain and range of the function $\left(\frac{f}{g}\right)(x)$.

$x \mid x \neq 0, \pm 3, x \in R$

$y \mid y \neq 0, \pm \frac{1}{3}, y \in R$.

3. Consider the functions $f(x) = \frac{3}{x+2}$ and $g(x) = x - 2$.

a) Show that $(g \circ f)(x) = \frac{-2x-1}{x+2}$.

$(g \circ f)(x) = g\left(\frac{3}{x+2}\right) = \frac{3}{x+2} - 2 = \frac{3}{x+2} - \frac{2(x+2)}{x+2} = \frac{3-2(x+2)}{(x+2)}$

$= \frac{3-2x-4}{x+2} = \frac{-2x-1}{x+2}$ ✓

b) Determine the equations of the asymptotes of the graph of $y = (g \circ f)(x)$.

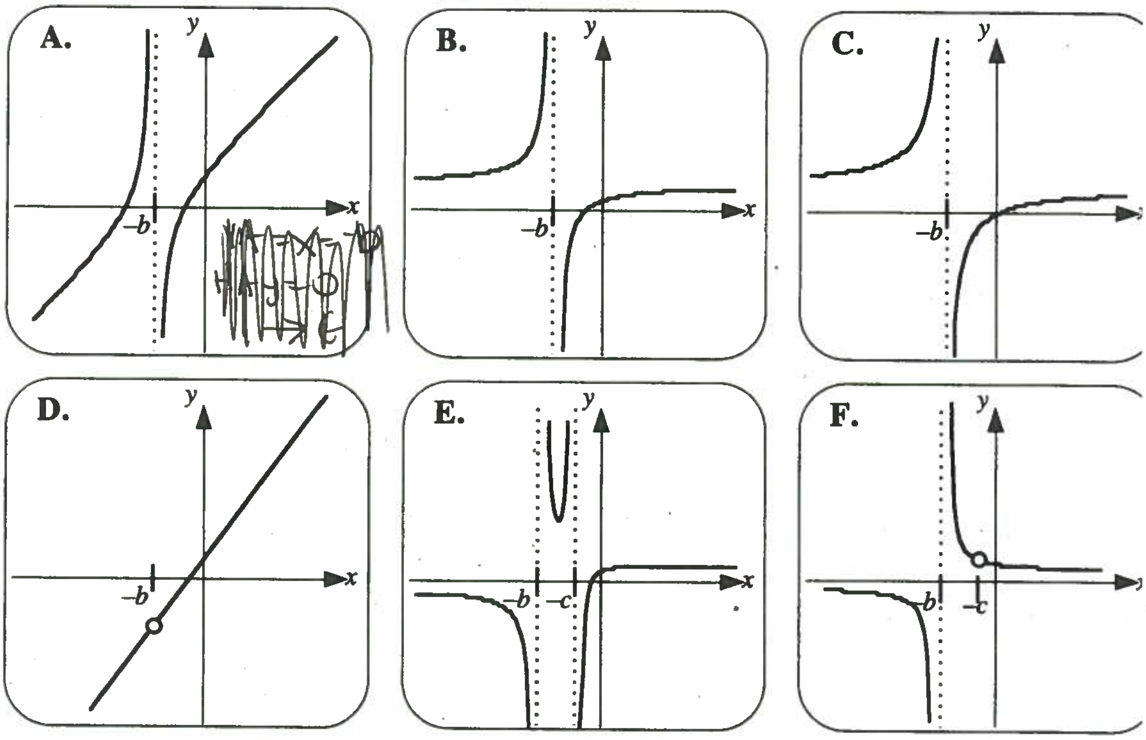
HA $y = -2$ $\quad$ VA $x = -2$

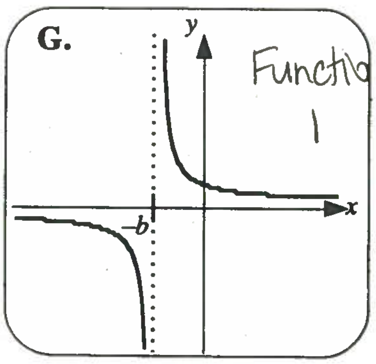
c) State the domain and range of the function $(g \circ f)(x)$.

$x \mid x \neq -2, x \in R$ $\quad$ $y \mid y \neq -2, y \in R$.

4. The equations of seven rational functions and the graphs of these functions are shown. If $a, b,$ and c are distinct natural numbers, match the set of rational functions to their grap and explain the reasoning.

Function 1	**Function 2**	**Function 3**	**Function 4**
$y=\frac{1}{x+b}$ VA $x=-b$ HA $y=0$	$y=\frac{x}{x+b}$	$y=\frac{x+a}{x+b}$	$y=\frac{(x+a)(x+c)}{x+b}$
Function 5	**Function 6**	**Function 7**	
$y=\frac{(x+a)(x+b)}{x+b}$	$y=\frac{x+a}{(x+b)(x+c)}$	$y=\frac{x+c}{(x+b)(x+c)}$	





Function 1
Function VA: $x=-b$
1 HA $y=0$ 1→G

Function 2+3
VA $x=-b$
HA $y=1$
Fun 2 → passes through origin+
Fun 3 does not so 2→C + 3→

Function 4+5
→no HA
Fun 4 → VA $x=-b$
Fun 5 → PD $x=-b$
4→A, 5→D

Functions 6+7
→2 discontinuities
Fun 6 → 2 VA so 6→E
Fun 7 → 1 VA + 1 pt of discontinuity 7→F

1→G
2→C
3→B
4→A
5→D
6→E
7→F

Multiple Choice

5. Which of the following functions has a point of discontinuity at (p, q)?

A. $y = \dfrac{(x-p)(x-q)}{x-p}$ $x - q,\ x \neq p$

B. $y = \dfrac{(x-p)(x-q)}{x-q}$ $x - p,\ x \neq q$

C. $y = \dfrac{(x-p)(x-p+q)}{x-p}$ $x - p + q,\ x \neq p$

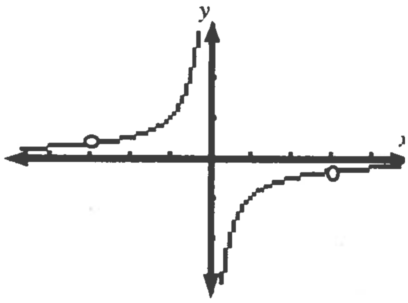
D. $y = \dfrac{(x-q)(x+p-q)}{x-q}$ $x + p - q,\ x \neq q$

If $x = p$ $y = p - q$

If $x = p$ $y = p - p + q = q$

Answer Key

1. a) horizontal asymptote $y = 0$. b) vertical asymptote $x = -1$, horizontal asymptote $y = -4$
   c) horizontal asymptote $y = -\frac{1}{3}$
   d) vertical asymptote $x = -1$, horizontal asymptote $y = 0$, point of discontinuity $\left(-4, -\frac{1}{3}\right)$.
   e) vertical asymptote $x = -1$, $x = -\frac{1}{3}$, horizontal asymptote $y = 2$.
   f) vertical asymptote $x = \frac{2}{3}$, horizontal asymptote $y = 2$, point of discontinuity $\left(-4, \frac{13}{7}\right)$.
2. a) $f(x) = (x-3)(x+3)$, $g(x) = x(3-x)(3+x)$
   b) $\left(\dfrac{f}{g}\right)(x) = -\dfrac{1}{x}, x \neq \pm 3$
   c) vertical asymptote $x = 0$, horizontal asymptote $y = 0$,
   points of discontinuity $\left(-3, \frac{1}{3}\right)$, $\left(3, -\frac{1}{3}\right)$.
   d) See graph at side.
   e) Domain: $\{x \mid x \neq 0, \pm 3, x \in R\}$
   Range: $\left\{y \mid y \neq 0, \pm\frac{1}{3}, y \in R\right\}$



3. b) vertical asymptote $x = -2$, horizontal asymptote $y = -2$.
   c) Domain: $\{x \mid x \neq -2, x \in R\}$ Range: $\{y \mid y \neq -2, y \in R\}$

4. Function 1 → G, Function 2 → C, Function 3 → B, Function 4 → A, Function 5 → D, Function 6 → E, Function 7 → F.

   - Function 1 has a vertical asymptote $x = -b$ and a horizontal asymptote $y = 0$, so Function 1 → G.
   - Functions 2 and 3 have a vertical asymptote $x = -b$ and a horizontal asymptote $y = 1$. Function 2 pass through the origin and Function 3 does not, so Function 2 → C and Function 3 → B.
   - Functions 4 and 5 have no horizontal asymptotes. Function 4 has a vertical asymptote $x = -b$, and Function 5 has a point of discontinuity at $x = -b$, so Function 4 → A and Function 5 → D.
   - Functions 6 and 7 have two discontinuities. Function 6 has two vertical asymptotes, so Function 6 → Function 7 has one vertical asymptote and one point of discontinuity, so Function 7 → F.

5. C